Construction of G-Hilbert schemes

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Abstract

One objective of this paper is to provide a reference for certain fundamental constructions in the theory of G-Hilbert schemes. In this paper we review, extend and develop this theory, we introduce a relative version of G-Hilbert schemes, consider the morphism G-Hilb $X \to X/G$ and vary the base scheme. This allows to construct the G-Hilbert scheme as a scheme over the quotient, we replace the assumption "quasiprojective" by the more natural assumption that a geometric quotient exists.

We also make observations on the McKay correspondence, we relate the stratification of the G-Hilbert scheme introduced by Ito-Nakamura to relative tangent spaces of the morphism G-Hilb $\mathbb{A}^n \to \mathbb{A}^n/G$.

Contents

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Introduction

The G-Hilbert scheme construction over a base scheme S in general forms under some assumptions an S-scheme $\text{G-Hilb}_S X$ for a given S-scheme X with operation over S of a finite group scheme G.

This construction, introduced in [ItNm96], [ItNm99], [Nm01] for $\mathbb{A}^n_{\mathbb{C}}$ or more generally for quasiprojective schemes over the base scheme $\operatorname{Spec}\mathbb{C}$, is motivated by the problem to resolve quotient singularities and to describe the properties of such resolutions. It has been made extensive use of it in works concerning the McKay correspondence with [BKR01] the most prominent one, see [Re97], [Re99] for introductions to this subject. It should be remarked, that at least two inequivalent definitions of a G-Hilbert scheme can be found in the literature: For us the G-Hilbert scheme is the moduli space of G-clusters on X. In the usual cases its underlying reduced subscheme has a component birational to the quotient X/G, which sometimes is taken to be the G-Hilbert scheme.

In this paper on the one hand we go through well known constructions – although they have been applied in several works, some of these foundational constructions seem not to be available in a form that can serve as a reference. Some of this material is contained in the thesis [Té04]. We formulate for the case of finite group schemes with cosemisimple Hopf algebra over arbitrary fields what has been done before for finite groups over $\mathbb{C}$. This becomes necessary with view toward generalizations of McKay correspondence to base schemes other than $\mathbb{C}$, the slightly more general case of non algebraically closed fields K of characteristic 0 and finite subgroup schemes $G \subset \mathrm{SL}(2, K)$ had been investigated in [Bl06].

As mentioned at the beginning, we will introduce a relative G-Hilbert scheme construction with respect to a morphism $X \to S$, X with G-operation over S. On the level of generality taken here, the group scheme G as well as the scheme S will be defined over a field K. We will construct a morphism of functors $\underline{\text{G-Hilb}}_S X \to \underline{X/G}$ and consider ways of changing the base scheme of G-Hilbert functors. In particular $\underline{\text{G-Hilb}}_S X$ can be considered as a X/G-functor and then coincides with the relative G-Hilbert functor $\underline{\text{G-Hilb}}_{X/G} X$.

This extended theory leads to some improvements and simplifications, for example the fibration of the G-Hilbert functor over the functor of the quotient leads to a representability proof of G-Hilbert functors for algebraic K-schemes X which replaces the assumption "quasiprojective" by the more natural condition that a geometric quotient $\pi : X \to X/G$, π affine, exists and works without the theorem on representability of n-point Hilbert functors.

Thinking about the G-Hilbert scheme as $\text{G-Hilb}_{X/G} X$, one can carry out the differential study for Quot schemes [Gr61, Section 5] in the equivariant setting to determine relative tangent spaces of the G-Hilbert scheme over X/G. There is a relation to the stratification introduced in [ItNm96], [ItNm99].

By the point of view taken here, the theory of G-Hilbert schemes is developed in the context of Quot schemes [Gr61], replacing categories of quasicoherent sheaves by categories of quasicoherent G-sheaves. Thus, it requires some theory of G-sheaves as summarized in subsection 1.2. We observe, that the group operation has the tendency to simplify things in comparision to the case of n-point Hilbert schemes, for example when constructing the morphism $\text{G-Hilb}_K X \to X/G$ and when proving representability. Further, sometimes the quotient X/G should be viewed as the prefered base scheme.

This paper only comprises the most fundamental constructions that already had become necessary in the study of the McKay correspondence. Extensions in several directions may be developed someday, for example:

- The quotient is functorial with respect to equivariant morphisms, the G-Hilbert scheme, being projective over it, has functorial properties similar to a $\operatorname{Proj} \mathscr{S}$. In this paper we only have considered the case $(G,X) \to (\{id\},Y)$, apart from equivariant open and closed embeddings, other situations of equivariant $(G,X) \to (H,Y)$ might become interesting.
- One may consider finite flat group schemes over more general base schemes than spectra of fields.
- One may construct partial resolutions in between the G-Hilbert scheme and the quotient.

We now summarize the contents in more detail:

Preparatory, we review definitions and results concerning Quot and Hilbert schemes. These schemes are defined by their functor of points to parametrize certain quotient sheaves resp. closed subschemes of a scheme X (quasi)projective over a base scheme S. There is a discrete quantity, the Hilbert polynomial, which leads to a decomposition into open and closed subfunctors corresponding to fixed Hilbert polynomials. That any of these components is represented by a (quasi)projective scheme is the representability theorem of Grothendieck. The reference for this is [Gr61], also see [Ko, Ch. I], [HL, Ch. 2.2]. Language and methods will be taken over to the theory of G-Hilbert schemes.
We then summarize some theory of G-sheaves for group schemes G starting with the definition of [Mu, GIT]. Since this paper is not the place to develop this whole theory we are content with stating the relevant results.

Working with categories of G-sheaves on a G-scheme X for a group scheme G over a base scheme S one can carry out the same Quot and Hilbert scheme construction as in the ordinary case in [Gr61]. These equivariant Quot schemes are closed subschemes of the original Quot schemes, they can equivalently be described as fixed point subschemes with respect to the natural G-operation.
Assume that $G = \operatorname{Spec} A$ is an affine group scheme over K, its Hopf algebra A being cosemisimple, and let G operate on X over S. The G-sheaf structures of the parametrised quotients allow to refine the decomposition by Hilbert polynomials. We are interested in the case of constant Hilbert polynomial, then one can specify the representations of the fibers to be of a certain isomorphism class. In these constructions G had not assumed to be finite, only affine with cosemisimple Hopf algebra, however, the case of primary interest for us arises for finite group schemes G over K: Taking the component of the equivariant Hilbert scheme with $|G|$ as (constant) Hilbert polynomial and the regular representation as representation on the fibers, this leads to the definition of the G-Hilbert scheme.
We review and extend the representability results of the G-Hilbert functor $\underline{\text{G-Hilb}}_S\, X$ for quasiprojective S-schemes X using the general representability theorem of [Gr61] in the case of Hilbert functors of points by showing that the G-Hilbert functor is an open and closed subfunctor of the equivariant Hilbert functor which is a closed subfunctor of the ordinary Hilbert functor of $|G|$ points.

The next main theme and main part of this paper is the relation of the G-Hilbert scheme to the quotient X/G. In this part we consider G-schemes X that are not neccessarily quasiprojective over S. We construct the natural morphism $\underline{\text{G-Hilb}}_S\, X \to \underline{X/G}$. More generally, any equivariant S-morphism $X \to Y$, Y with trivial G-operation, induces a morphism $\underline{\text{G-Hilb}}_S\, X \to \underline{Y}$. This can be used to relate the Hilbert functor $\underline{\text{G-Hilb}}_S\, X$ and the relative Hilbert functor $\underline{\text{G-Hilb}}_{X/G}\, X$, because it allows to consider $\underline{\text{G-Hilb}}_S\, X$ as a functor over X/G. We pursue the general idea to vary the base scheme of G-Hilbert functors and give some applications.
For finite $X \to S$ Hilbert schemes of points and G-Hilbert schemes can be constructed directly as projective schemes over S by showing that a natural embedding into a Grassmannian is closed. Applied to the relative G-Hilbert functor $\underline{\text{G-Hilb}}_{X/G}\, X$ for an affine geometric quotient morphism $X \to X/G$ of an algebraic K-scheme X this shows the existence of the projective X/G-scheme $\text{G-Hilb}_{X/G}\, X$. The earlier investigations about base changes then imply that $\underline{\text{G-Hilb}}_K\, X$ is representable and that there is an isomorphism of K-schemes $\text{G-Hilb}_K\, X \cong {}_K(\text{G-Hilb}_{X/G}\, X)$, which identifies $\tau : \text{G-Hilb}_K\, X \to X/G$ with the structure morphism of $\text{G-Hilb}_{X/G}\, X$. In particular, one sees that the morphism τ is projective.
We make some remarks on the case of free operation. Here $\tau : \text{G-Hilb}_K\, X \to X/G$ is an isomorphism. Thus, if for an irreducible variety X the operation is free on an open dense subscheme, then $(\text{G-Hilb}_K\, X)_{\text{red}}$ has a unique irreducible component birational to X/G.

We carry out the differential study for Quot schemes [Gr61, Section 5] (see also [Ko], [HL]) in the equivariant setting. Since this generalization is straightforward, we state the results only. Also, the same method is known and used to determine tangent spaces of G-Hilbert schemes over base fields. But this method as well applies to relative G-Hilbert schemes and allows to determine their sheaf of relative differentials and their relative tangent spaces. We observe, that relative tangent spaces of $\text{G-Hilb}_{\mathbb{C}}\, \mathbb{A}^2_{\mathbb{C}}$ over $\mathbb{A}^2_{\mathbb{C}}/G$ are related to the stratification introduced in [ItNm96], [ItNm99].

Notations:
- In general we write a lower index for base extensions, for example if X, T are S-schemes then X_T denotes the T-scheme $X \times_S T$ or if V is a representation over a field K then V_L denotes the representation $V \otimes_K L$ over the extension field L.
- Likewise for morphisms of schemes: If $\varphi : X \to Y$ is a morphism of S-schemes and $T \to S$ an S-scheme, then write φ_T for the morphism $\varphi \times id_T : X_T \to Y_T$ of T-schemes.
- Let $\psi : T \to S$ be a morphism and $\mathscr{F}$ an $\mathcal{O}_S$-module. Then write $\mathscr{F}_T$ for the $\mathcal{O}_T$-module $\psi^*\mathscr{F}$ and put $\alpha_T = \psi^*\alpha$ if $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_S$-modules.
- More generally let X be an S-scheme, $\mathscr{F}$ an $\mathcal{O}_X$-module and let $\psi : T \to S$ a morphism. Then write $\mathscr{F}_T$ for the $\mathcal{O}_{X_T}$-module $\psi_X^*\mathscr{F}$ and α_T for $\psi_X^*\alpha$ if $\alpha : \mathscr{F} \to \mathscr{G}$ is a morphism of $\mathcal{O}_X$-modules.
- Sometimes we will underline a functor to distinguish it in case of representability from the corresponding scheme, in general the functor corresponding to a scheme will be denoted by the same symbol underlined.

1 Preliminaries: Quot schemes and G-sheaves

1.1 Quot and Hilbert schemes

Let S be a scheme (later mostly assumed to be noetherian) and $f : X \to S$ an S-scheme.

Quotient sheaves. Let $\mathscr{F}$ be a quasicoherent $\mathcal{O}_X$-module. By a quotient sheaf of $\mathscr{F}$ we mean an exact sequence $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0$ of quasicoherent $\mathcal{O}_X$-modules with $\mathscr{H}, \mathscr{G}$ specified up to isomorphism, that is either a quasicoherent subsheaf $\mathscr{H} \subseteq \mathscr{F}$ or an equivalence class $[\mathscr{F} \to \mathscr{G}]$ of quasicoherent quotients, where two quotients are defined to be equivalent, if their kernels coincide. Also write $[0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0]$ for the corresponding equivalence class.

Quot and Hilbert functors. For a quasicoherent $\mathcal{O}_X$-module $\mathscr{F}$ the Quot functor for $\mathscr{F}$ on X over S is the functor

$$\begin{array}{rccl} \underline{\mathrm{Quot}}_{\mathscr{F}/X/S} : & (S\textit{-schemes})^\circ & \to & (\textit{sets}) \\ & T & \mapsto & \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T \end{array} \right\} \end{array}$$

where for a morphism of S-schemes $\varphi : T' \to T$ the map $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(\varphi) : \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T) \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T')$ is defined by application of φ_X^*, i.e. $[\mathscr{F}_T \to \mathscr{G}] \mapsto [\mathscr{F}_{T'} \to \varphi_X^* \mathscr{G}]$.
The Hilbert functor arises as the special case $\mathscr{F} = \mathcal{O}_X$:

$$\begin{array}{rccl} \underline{\mathrm{Hilb}}_{X/S} : & (S\textit{-schemes})^\circ & \to & (\textit{sets}) \\ & T & \mapsto & \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \\ \mathcal{O}_Z \textit{ flat over } T \end{array} \right\} \end{array}$$

Later, we sometimes only mention the closed subscheme $Z \subseteq X_T$ and write $Z \in \underline{\mathrm{Hilb}}_{X/S}(T)$ instead of the whole quotient.

Decomposition via Hilbert polynomials. In the following assume that S is noetherian, that $f : X \to S$ is projective with $\mathcal{O}_X(1)$ a very ample line bundle relative to f and that $\mathscr{F}$ is coherent.
Then for any locally noetherian S-scheme $T \to S$ the $\mathcal{O}_{X_T}$-module $\mathscr{F}_T$ is coherent and so are its quotients. The morphism $f_T : X_T \to T$ is projective with very ample line bundle $\mathcal{O}_{X_T}(1)$, for any point $t \in T$ with residue field $\kappa(t)$ one has the projective $\kappa(t)$-scheme $f_t : X_t \to \operatorname{Spec} \kappa(t)$.
For a coherent $\mathcal{O}_{X_T}$-module $\mathscr{G}$ there is for any point $t \in T$ the Hilbert polynomial of the coherent $\mathcal{O}_{X_t}$-module $\mathscr{G}_t$: The Euler-Poincaré characteristic of the twisted sheaves $\mathscr{G}_t(n) = \mathscr{G}_t \otimes_{\mathcal{O}_{X_t}} \mathcal{O}_{X_t}(n)$

$$\chi(\mathscr{G}_t(n)) = \textstyle\sum_{i \geq 0} (-1)^i h^i(X_t, \mathscr{G}_t(n))$$

is a polynomial in n ([EGA, III, (1), (2.5.3)]), that is $\chi(\mathscr{G}_t(n)) = p_{\mathscr{G}_t}(n)$ for some $p_{\mathscr{G}_t} \in \mathbb{Q}[z]$. $p_{\mathscr{G}_t}$ is called the Hilbert polynomial of $\mathscr{G}_t$.
We restrict the functors $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ and $\underline{\mathrm{Hilb}}_{X/S}$ to the category of locally noetherian S-schemes and define subfunctors $\underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S} : (\textit{loc. noeth. } S\textit{-schemes})^\circ \to (\textit{sets})$ by

$$\underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T, \textit{ fibers of } \mathscr{G} \textit{ have Hilbert polynomial } p \end{array} \right\}$$

and similarly $\underline{\mathrm{Hilb}}^p_{X/S}$. These are subfunctors, since the Hilbert polynomial is invariant under pull-backs. Note, that in general the assignment of a component to a given polynomial depends on the choice of isomorphism class of a very ample line bundle $\mathcal{O}_X(1)$.

If $\mathscr{G}$ is flat over T, then the Hilbert polynomial of $\mathscr{G}$ is locally constant on T ([EGA, III, (2), (7.9.11)]). It follows, that these subfunctors are open and closed. Further, they cover the original functor, thus there are decompositions

$$\underline{\mathrm{Quot}}_{\mathscr{F}/X/S} = \coprod_p \underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}, \qquad \underline{\mathrm{Hilb}}_{X/S} = \coprod_p \underline{\mathrm{Hilb}}^p_{X/S}$$

Representability. The question of representability of the Quot and Hilbert functors reduces to that of the subfunctors for fixed Hilbert polynomials. For these there is the following theorem:

Theorem 1.1 ([Gr61]). *Let S be a noetherian scheme, $f : X \to S$ a projective morphism, $\mathcal{O}_X(1)$ a very ample line bundle relative to f and $\mathscr{F}$ a coherent $\mathcal{O}_X$-module. Then the functor $\underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}$: (loc. noeth. S-schemes)$^\circ \to$ (sets) is representable by a projective S-scheme.* □

This means, that there exists a projective S-scheme $\mathrm{Quot}^p_{\mathscr{F}/X/S}$ (unique up to isomorphism) with an isomorphism $\mathrm{Mor}_S(\,\cdot\,, \mathrm{Quot}^p_{\mathscr{F}/X/S}) \cong \underline{\mathrm{Quot}}^p_{\mathscr{F}/X/S}$. The morphism id_Q, where $Q := \mathrm{Quot}^p_{\mathscr{F}/X/S}$, corresponds to a universal quotient $[0 \to \mathscr{H} \to \mathscr{F}_Q \to \mathscr{G} \to 0]$ on X_Q, which determines the above isomorphism.

Base change. Let $\alpha : S' \to S$ be a noetherian S-scheme. Then $\mathscr{F}' := \mathscr{F}_{S'}$ is coherent on $X' := X_{S'}$ and the functor $\underline{\mathrm{Quot}}_{\mathscr{F}'/X'/S'}$ is the restriction of $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ to the category of S'-schemes. In this situation one has the following (see also [EGA1, 0, (1.3.10)]): If $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ is represented by Q with universal quotient $[\mathscr{F}_Q \to \mathscr{G}]$, then $\underline{\mathrm{Quot}}_{\mathscr{F}'/X'/S'}$ is represented by $Q' = Q_{S'}$ with universal quotient $[\mathscr{F}_{Q'} \to \mathscr{G}_{Q'}]$.

Grassmannian. The Grassmannian functor arises in the special case $X = S$. For any scheme S, a quasicoherent $\mathcal{O}_S$-module $\mathscr{F}$ and $n \in \mathbb{N}$ define

$$\underline{\mathrm{Grass}}^n_S(\mathscr{F})(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } T, \\ \mathscr{G} \textit{ locally free of rank } n \end{array} \right\}$$

The proof of representability of general Quot functors uses the representability of Grassmannian functors – it is shown that any Quot functor occurs as a closed subfunctor of a Grassmannian functor. The representability theorem for Grassmannian functors reads as follows:

Theorem 1.2 ([EGA1, I, (9.7.4)]). *For a scheme S, a quasicoherent $\mathcal{O}_S$-module $\mathscr{F}$ and $n \in \mathbb{N}$ the Grassmannian functor $\underline{\mathrm{Grass}}^n_S(\mathscr{F})$ is representable.* □

The Plücker embedding $\mathrm{Grass}^n_S(\mathscr{F}) \to \mathbb{P}_S(\bigwedge^n \mathscr{F})$ shows that for $\mathscr{F}$ quasicoherent of finite type $\mathrm{Grass}^n_S(\mathscr{F})$ is projective over S.

Closed subschemes. If $i : Y \to X$ is a closed embedding over S, then (in case of representability) there is the closed embedding of Quot schemes

$$\mathrm{Quot}_{i^*\mathscr{F}/Y/S} \cong \mathrm{Quot}_{i_*i^*\mathscr{F}/X/S} \to \mathrm{Quot}_{\mathscr{F}/X/S}$$

for example $\mathrm{Hilb}_{Y/S} \to \mathrm{Hilb}_{X/S}$.

Open subschemes. For an open subscheme $j : U \to X$ over S define

$$\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T,\ \mathrm{supp}(\mathscr{G}) \subseteq U_T \end{array} \right\}$$

This defines an open subfunctor, so (in case of representability) one has the open embedding

$$\mathrm{Quot}_{\mathscr{F}/X/S}|_U \to \mathrm{Quot}_{\mathscr{F}/X/S}$$

Quasi-projective schemes. For a quasiprojective S-scheme U and $\mathscr{E}$ coherent on U define the following variant (which coincides with the original Quot functor for projective U)

$$\underline{\mathrm{Quot}}_{\mathscr{E}/U/S}(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{H} \to \mathscr{E}_T \to \mathscr{G} \to 0] \textit{ on } U_T, \\ \mathscr{G} \textit{ flat over } T, \ \mathrm{supp}(\mathscr{G}) \textit{ proper over } T \end{array} \right\}$$

After choice of an open embedding over S into a projective S-scheme X and of a coherent prolongation of $\mathscr{E}$, that is a coherent $\mathcal{O}_X$-module $\mathscr{F}$ such that $\mathscr{F}|_U = \mathscr{E}$ (see [EGA1, I, (6.9.8)]), one has an open embedding

$$\underline{\mathrm{Quot}}_{\mathscr{E}/U/S} \cong \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}|_U \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$$

This extends the construction of Quot schemes to quasiprojective S-schemes, in this case the components $\underline{\mathrm{Quot}}^p_{\mathscr{E}/U/S}$ are representable by quasiprojective S-schemes.

Hilbert scheme of n points. Let X be quasiprojective over the noetherian scheme S. The component $\mathrm{Hilb}^n_{X/S}$ of $\mathrm{Hilb}_{X/S}$ for constant Hilbert polynomials $n \in \mathbb{N}$ (it is independent of choice of $\mathcal{O}_X(1)$) is called the Hilbert scheme of n points, we also write $\mathrm{Hilb}^n_S X$ for this scheme. Constant Hilbert polynomial means, that one parametrises 0-dimensional subschemes [EGA, IV, (2), (5.3.1)]. For these the Euler-Poincaré characteristic reduces to h^0. One may rewrite the corresponding functor as

$$\underline{\mathrm{Hilb}}^n_S X(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \ Z \textit{ flat and} \\ \textit{proper over } T, \ Z_t \ 0\textit{-dimensional with } h^0(Z_t, \mathcal{O}_{Z_t}) = n \textit{ for } t \in T \end{array} \right\} \quad (1)$$

The condition "*Z flat, proper over T, Z_t 0-dimensional with $h^0(Z_t, \mathcal{O}_{Z_t}) = n$ for $t \in T$*" implies
- $H^i(Z_t, \mathcal{O}_{Z_t}) = 0$ for $i > 0$
- $f_{T*}\mathcal{O}_Z$ is locally free of rank n
- the canonical homomorphisms $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathcal{O}_Z \to H^0(Z_t, \mathcal{O}_{Z_t})$ for $t \in T$ are isomorphisms

by the theorem on cohomology and base change [Ha, AG, Thm. 12.11], [EGA, III (2), (7.9.9)], [Mu, GIT, Ch. 0, §5, p. 19]. Moreover, in this situation Z is finite over T by [EGA, III (1), (4.4.2)] or [EGA, IV (3), (8.11.1)] (of course the above cohomology and base change property then also follows). Similar statements are true for the Quot functors $\underline{\mathrm{Quot}}^n_{\mathscr{F}/X/S}$.

Hilbert scheme of 1 point. Under not restrictive assumptions the Hilbert functor $\underline{\mathrm{Hilb}}^1_S X$ as defined in equation (1) is represented by the original S-scheme X, we show that there are natural bijections $\underline{\mathrm{Hilb}}^1_S X(T) \leftrightarrow \Gamma(X_T/T) = X(T)$ (see also [AlKl80, Lemma (8.7), p. 108]):

Proposition 1.3. *Let $f : X \to S$ be locally of finite type over the noetherian scheme S. Then there is a morphism $\underline{\mathrm{Hilb}}^1_S X \to \underline{X}$ taking an element $Z \in \underline{\mathrm{Hilb}}^1_S(T)$ to the morphism $T \cong Z \subseteq X_T \to X$. If X is separated over S then this is an isomorphism and determined by $\Delta \leftrightarrow id_X$, $\Delta \subseteq X \times_S X$ the diagonal.*

Proof. For the first statement, the main point is to show that for any $Z \subseteq X_T$, T a locally noetherian S-scheme, defining an element of $\underline{\mathrm{Hilb}}^1_S X(T)$ the restriction $f_T|_Z : Z \to T$ is an isomorphism. Z is flat, proper and quasifinite over T, then by [EGA, III (1), (4.4.2)] or [EGA, IV (3), (8.11.1)] Z is finite over T, so $f_T|_Z : Z \to T$ corresponds to a homomorphism $\mathcal{O}_T \to \mathscr{A}$ of locally free $\mathcal{O}_T$-algebras of finite rank. This is an isomorphism, because $\kappa(t) \to \kappa(t) \otimes_{\mathcal{O}_T} \mathscr{A}$ for any $t \in T$ is an isomorphism. The maps $\underline{\mathrm{Hilb}}^1_S X(T) \to \Gamma(X_T/T) = X(T)$ take an element $Z \in \underline{\mathrm{Hilb}}^1_S X(T)$ to the section $T \to Z \subseteq X_T$ inverse to $f_T|_Z : Z \to T$.
If X is separated over S, then there is an inverse morphism $\underline{X} \to \underline{\mathrm{Hilb}}^1_S X$ given by $id_X \mapsto \Delta$ consisting of maps $\underline{X}(T) \to \underline{\mathrm{Hilb}}^1_S X(T)$, $\varphi \mapsto \Gamma_\varphi$ for S-schemes T. Here $\Gamma_\varphi = (id \times \varphi)^{-1}(\Delta) \subseteq X \times_S T$ is the graph of φ, it also arises as the closed subscheme corresponding to the closed embedding $(\varphi, id_T) : T \to X \times_S T$ (closed embedding, because X is separated over S). $\square$

1.2 G-sheaves

Group schemes. A group scheme over S is an S-scheme $p : G \to S$ with morphisms e, m, i over S, $e : S \to G$ the unit, $m : G \times_S G \to G$ the multiplication, $i : G \to G$ the inverse, satisfying the usual axioms.

Operations of group schemes. Let G be a group scheme over S. A G-scheme X over S is an S-scheme X with an operation of G over S, that is a morphism of S-schemes $s_X : G \times_S X \to X$ satisfying the usual conditions of a group operation. It can be viewed in different ways:

Remark 1.4. A group scheme operation $s_X : G \times_S X \to X$ over S is equivalent to operations $G(T) \times_{S(T)} X(T) = (G \times_S X)(T) \to X(T)$ of the groups of T-valued points $G(T)$ on the sets $X(T)$ for S-schemes T that are functorial in T.
For a fixed $g \in G(T)$ one has for T-schemes T' bijections $X(T') \to X(T')$ functorial in T' and thus an isomorphism $\varphi_g : X_T \to X_T$ of T-schemes. Note that φ_g is also given by the composition

$$X_T = T \times_T X_T \xrightarrow{g \times id_{X_T}} G_T \times_T X_T \xrightarrow{s_{X_T}} X_T$$

The map $g \mapsto \varphi_g$ is a group homomorphism of $G(T)$ to the automorphism group of X_T over T. Later, the symbol g will be used for $\varphi_g : X_T \to X_T$ as well.

***G*-sheaves.** Let $p : G \to S$ be a group scheme over a scheme S, let X be a G-scheme over S with operation $s_X : G \times_S X \to X$. A (quasicoherent, coherent) G-sheaf on X is a (quasicoherent, coherent) $\mathcal{O}_X$-module $\mathscr{F}$ with an isomorphism

$$\lambda^{\mathscr{F}} : s_X^* \mathscr{F} \xrightarrow{\sim} p_X^* \mathscr{F}$$

of $\mathcal{O}_{G \times_S X}$-modules satisfying

(i) The restriction of $\lambda^{\mathscr{F}}$ to the unit in G_X is the identity, i.e. the following diagram commutes:

$$\begin{tikzcd} e_X^* s_X^* \mathscr{F} \arrow[r, "e_X^* \lambda^{\mathscr{F}}"] \arrow[d, equal] & e_X^* p_X^* \mathscr{F} \arrow[d, equal] \\ \mathscr{F} \arrow[r, "id_{\mathscr{F}}"] & \mathscr{F} \end{tikzcd}$$

(ii) $(m \times id_X)^* \lambda^{\mathscr{F}} = \mathrm{pr}_{23}^* \lambda^{\mathscr{F}} \circ (id_G \times s_X)^* \lambda^{\mathscr{F}}$ on $G \times_S G \times_S X$, where $\mathrm{pr}_{23} : G \times_S G \times_S X \to G \times_S X$ is the projection to the factors 2 and 3.

This definition is taken from [Mu, GIT], in addition we will introduce another equivalent definition similar to the one used in [BKR01].

Construction 1.5. For an S-scheme T and $g \in G(T) = G_T(T)$ there is the diagram

$$\begin{tikzcd} & X_T = T \times_T X_T = X_T \arrow[dl, "g"'] \arrow[d, "g \times id_{X_T}"] \arrow[dr, "id_{X_T}"] & \\ X_T & G_T \times_T X_T \arrow[l, "s_{X_T}"] \arrow[r, "p_{X_T}"'] & X_T \end{tikzcd}$$

Pulling back $(\lambda^{\mathscr{F}})_T : s_{X_T}^* \mathscr{F}_T \to p_{X_T}^* \mathscr{F}_T$ from $G_T \times_T X_T$ to X_T by $g \times id_{X_T}$ leads to an isomorphism $\lambda_g^{\mathscr{F}_T} : g^* \mathscr{F}_T \to \mathscr{F}_T$.

Proposition 1.6. *Let $\mathscr{F}$ be an $\mathcal{O}_X$-module on a G-scheme X over S. Then a G-sheaf structure on $\mathscr{F}$ is equivalent to the following data: For any S-scheme T and any T-valued point $g \in G(T) = G_T(T)$ an isomorphism $\lambda_g^{\mathscr{F}_T} : g^* \mathscr{F}_T \to \mathscr{F}_T$ of $\mathcal{O}_{X_T}$-modules such that $\lambda_{g_{T'}}^{\mathscr{F}_{T'}} = (\lambda_g^{\mathscr{F}_T})_{T'}$*

for T-schemes T' and the properties

(i) $\lambda_{e_T}^{\mathscr{F}_T} = id_{\mathscr{F}_T}$ *for* $e_T : T \to G_T$ *the identity of* $G(T) = G_T(T)$

(ii) $\lambda_{hg}^{\mathscr{F}_T} = \lambda_g^{\mathscr{F}_T} \circ g^*\lambda_h^{\mathscr{F}_T}$ *for* $g, h \in G(T)$

are satisfied. The correspondence is given by construction 1.5 and by specialization to $T = G$*,* $g = id_G$*, that is* $\lambda^{\mathscr{F}} = \lambda_{id_G}^{\mathscr{F}_G}$. □

Here condition (ii) of the definition arises as $\lambda_{p_1 \cdot p_2}^{\mathscr{F}_{G\times_S G}} = \lambda_{p_2}^{\mathscr{F}_{G\times_S G}} \circ p_2^*\lambda_{p_1}^{\mathscr{F}_{G\times_S G}}$, where $p_1, p_2 \in G(G \times_S G)$ are the projections. If G is flat, étale, ... over S, then it suffices to consider flat, étale, ... S-schemes T, for G a finite group regarded as a discrete group scheme over K it suffices to consider its K-valued points.

Remark 1.7. The requirement for $\lambda^{\mathscr{F}}$ and for the $\lambda_g^{\mathscr{F}_T}$ to be isomorphisms is not necessary, it follows from conditions (i) and (ii).

Example 1.8. For a G-scheme X over S the structure sheaf $\mathcal{O}_X$ and the sheaf of differentials $\Omega_{X/S}$ have natural G-sheaf structures.

***G*-subsheaves.** Let G be a flat group scheme over S and let $\mathscr{F}$ be a G-sheaf on a G-scheme X. A subsheaf $\mathscr{F}' \subseteq \mathscr{F}$ is called G-stable or a G-subsheaf if

$$\lambda^{\mathscr{F}}(s_X^*\mathscr{F}') \subseteq p_X^*\mathscr{F}'$$

Remark 1.9. (G-sheaf structure on a G-stable subsheaf). Let $\mathscr{F}' \subseteq \mathscr{F}$ be a G-subsheaf. Then the restriction $\lambda^{\mathscr{F}}|_{s_X^*\mathscr{F}'}$ defines a G-sheaf structure on $\mathscr{F}'$: The conditions (i) and (ii) of the definition of G-sheaves remain valid for $\lambda^{\mathscr{F}'} := \lambda^{\mathscr{F}}|_{s_X^*\mathscr{F}'}$ and by remark 1.7 $\lambda^{\mathscr{F}'}$ is an isomorphism.

Remark 1.10. Let $\mathscr{F}$ be a G-sheaf on X and $\mathscr{F}' \subseteq \mathscr{F}$ be a subsheaf. Then $\mathscr{F}'$ is a G-subsheaf if and only if $\lambda_g^{\mathscr{F}_T} : g^*\mathscr{F}_T \to \mathscr{F}_T$ restricts to $g^*\mathscr{F}'_T \to \mathscr{F}'_T$ for all flat S-schemes T and $g \in G(T)$.

Remark 1.11. (Quotients of G-sheaves). Let $\mathscr{F}' \subseteq \mathscr{F}$ be a G-subsheaf. Then $\mathscr{F}/\mathscr{F}'$ has a natural G-sheaf structure induced from $\mathscr{F}$.

Equivariant homomorphisms. Denote by $\mathrm{Hom}_X^G(\mathscr{F}, \mathscr{G})$ the set of those $\varphi \in \mathrm{Hom}_X(\mathscr{F}, \mathscr{G})$ for which the diagram

$$\begin{tikzcd} s_X^*\mathscr{F} \arrow[r, "s_X^*\varphi"] \arrow[d, "\lambda^{\mathscr{F}}"'] & s_X^*\mathscr{G} \arrow[d, "\lambda^{\mathscr{G}}"] \\ p_X^*\mathscr{F} \arrow[r, "p_X^*\varphi"] & p_X^*\mathscr{G} \end{tikzcd}$$

commutes. For a flat group scheme G and $\varphi \in \mathrm{Hom}_X^G(\mathscr{F}, \mathscr{G})$ the kernel and cokernel of φ have natural G-sheaf structures. One can form categories of G-sheaves like $\mathrm{Mod}^G(X)$, $\mathrm{Qcoh}^G(X)$ or $\mathrm{Coh}^G(X)$ with G-sheaves as objects and equivariant homomorphisms as morphisms, these are abelian.

Constructions, adjunctions and natural isomorphisms.
(1) The bifunctors $\otimes, \mathscr{H}om$ for sheaves have analogues for G-sheaves: For G-sheaves $\mathscr{F}, \mathscr{G}$ the tensor product $\mathscr{F} \otimes_{\mathcal{O}_X} \mathscr{G}$ has a natural G-sheaf structure. For quasicoherent G-sheaves $\mathscr{F}, \mathscr{G}$ with $\mathscr{F}$ finitely presented the sheaf $\mathscr{H}om_{\mathcal{O}_X}(\mathscr{F}, \mathscr{G})$ has a natural G-sheaf structure provided that the formation of $\mathscr{H}om$ commutes with base extension by G – this is satisfied if G is flat over S.

(2) Let $f : X \to Y$ be an equivariant morphism of G-schemes over S. Then one has functors f^*, f_* for the corresponding categories of G-sheaves that are the usual ones on the underlying categories of sheaves, where for f_* we will assume commutation with base extensions by G – for example this is satisfied for quasicoherent sheaves if f is affine or if f is quasicompact and quasiseparated and G is flat over S. There is the adjunction (f^*, f_*) extending the usual adjunction (f^*, f_*).
(3) Natural isomorphisms connecting compositions of these functors such as $f^*\mathscr{F} \otimes_{\mathcal{O}_X} f^*\mathscr{G} \cong f^*(\mathscr{F} \otimes_{\mathcal{O}_Y} \mathscr{G})$ or $f^*\mathscr{H}om_{\mathcal{O}_Y}(\mathscr{F}, \mathscr{G}) \cong \mathscr{H}om_{\mathcal{O}_X}(f^*\mathscr{F}, f^*\mathscr{G})$ for sheaves lead to isomorphisms for G-sheaves.

Comodules. Let $G = \operatorname{Spec}_S \mathscr{A}$ be an affine group scheme over S. $\mathscr{A}$ is a sheaf of $\mathcal{O}_S$-Hopf algebras. For an $\mathcal{O}_X$-module $\mathscr{F}$ on an S-scheme X one has the notion of an $\mathscr{A}$-comodule structure, that is a homomorphism of $\mathcal{O}_X$-modules

$$\mathscr{F} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{F}$$

satisfying the usual conditions of a comodule (see for example [Sw, Ch. II]).

Proposition 1.12. *Let X be a G-scheme with trivial operation. Then for an $\mathcal{O}_X$-module $\mathscr{F}$ the following data are equivalent:*

(a) *A G-sheaf structure on $\mathscr{F}$.*

(b) *An $\mathscr{A}$-comodule structure $\varrho : \mathscr{F} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{F}$.* □

Further, notions such as homomorphisms, subsheaves, etc. for comodules correspond to that for G-sheaves. For a G-sheaf $\mathscr{F}$ with G-sheaf structure equivalent to the $\mathscr{A}$-comodule structure $\varrho : \mathscr{F} \to \mathscr{A} \otimes_{\mathcal{O}_S} \mathscr{F}$ define the subsheaf of invariants $\mathscr{F}^G \subseteq \mathscr{F}$ by $\mathscr{F}^G(U) := \{f \in \mathscr{F}(U) \,|\, \varrho(f) = 1 \otimes f\}$ for open $U \subseteq X$.

Decomposition. For an affine group scheme $G = \operatorname{Spec}_S \mathscr{A}$ a G-sheaf on a G-scheme X with trivial G-operation can be decomposed as a comodule according to a direct sum decomposition of $\mathscr{A}$ into subcoalgebras.

Proposition 1.13. *Let $\mathscr{A}$ be an $\mathcal{O}_S$-coalgebra which decomposes into a direct sum of $\mathcal{O}_S$-subcoalgebras $\mathscr{A} = \bigoplus_i \mathscr{A}_i$. Then any $\mathscr{A}$-comodule $\mathscr{F}$ on X has a decomposition*

$$\mathscr{F} = \bigoplus\nolimits_i \mathscr{F}_i$$

into $\mathscr{A}$-subcomodules $\mathscr{F}_i$ whose comodule structure reduces to that of an $\mathscr{A}_i$-comodule. □

For simplicity we assume G finite over $S = \operatorname{Spec} K$, K a field, then $G = \operatorname{Spec}_K A$ for a finite dimensional Hopf-algebra A over K. Assume that A is cosemisimple (see [Sw, Ch. XIV]) and let $A = \bigoplus_i A_i$ be its decomposition into simple subcoalgebras. Then any G-sheaf $\mathscr{F}$ on X has an isotypical decomposition

$$\mathscr{F} = \bigoplus\nolimits_i \mathscr{F}_i$$

into G-subsheaves $\mathscr{F}_i$ whose G-sheaf structure is equivalent to an A_i-comodule structure. In particular, the G-invariant part $\mathscr{F}^G$ of $\mathscr{F}$ is the direct summand for the subcoalgebra corresponding to the trivial representation.

Representations. A G-sheaf over the spectrum of a field is also called a representation. Let $G = \operatorname{Spec}_K A$ be affine over K, then by proposition 1.12 this is the same as an A-comodule structure. Assume that G is finite over K, let $KG = A^\vee$ be the K-algebra dual to the coalgebra A (for discrete group schemes this is the group algebra). Dualisation of an A-comodule V over K leads to a KG-module $V^\vee$ and thereby to the more ordinary notion of a representation. Assume that A is cosemisimple, then the isotypical decomposition of an A-comodule V over K is the usual isotypical decomposition for representations.

2 Equivariant Quot schemes and the G-Hilbert scheme

2.1 Equivariant Quot and Hilbert schemes

Let G be a group scheme flat over a scheme S.

By a quotient G-sheaf of a quasicoherent G-sheaf $\mathscr{F}$ we mean an exact sequence $0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0$ in the category of G-equivariant quasicoherent sheaves with $\mathscr{H}, \mathscr{G}$ specified up to isomorphism or equivalently a quotient of $\mathscr{F}$ by a G-subsheaf, again write $[0 \to \mathscr{H} \to \mathscr{F} \to \mathscr{G} \to 0]$ or sometimes $[\mathscr{F} \to \mathscr{G}]$ for the corresponding equivalence class.

For a G-scheme X over S and a quasicoherent G-sheaf $\mathscr{F}$ on X define the equivariant Quot functor by

$$\underline{\mathrm{Quot}}^G_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T \end{array} \right\}$$

In particular, the G-sheaf $\mathcal{O}_X$ gives rise to the equivariant Hilbert functor

$$\underline{\mathrm{Hilb}}^G_{X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \\ \mathcal{O}_Z \textit{ flat over } T \end{array} \right\}$$

One also may consider the original Quot functors, where it is not assumed that a subsheaf $\mathscr{H} \subseteq \mathscr{F}_T$ is a G-subsheaf. Here the group operation on X and the G-sheaf structure of $\mathscr{F}$ induce an operation of G on the corresponding Quot schemes as follows:
For any S-scheme T a T-valued point $g \in G(T)$ defines an automorphism $g: X_T \to X_T$ (see remark 1.4), we define maps

$$\begin{array}{ccc} G(T) \times_S \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T) & \to & \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}(T) \\ (g, [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]) & \mapsto & [0 \to g_*\mathscr{H} \to \mathscr{F}_T \to g_*\mathscr{G} \to 0] \end{array}$$

by applying g_* and using the isomorphism $\mathscr{F}_T \to g_*\mathscr{F}_T$ coming from the G-sheaf structure of $\mathscr{F}$. These maps are group operations functorial in T, so they form an operation $\underline{G} \times_S \underline{\mathrm{Quot}}_{\mathscr{F}/X/S} \to \underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ of the group functor $\underline{G}$ on the Quot functor $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ and in case of representability an operation of the group scheme G on the scheme $\mathrm{Quot}_{\mathscr{F}/X/S}$ over S.

For any group scheme G operating on a scheme Y over S there is the notion of the fixed point subfunctor $\underline{Y}^G \subseteq \underline{Y}$ defined by (see also [DG, II, §1, Def. 3.4])

$$\underline{Y}^G(T) := \{x \in Y(T) \mid \textit{for all } T\textit{-schemes } T' \to T \textit{ and } g \in G(T'):\ gx_{T'} = x_{T'}\}$$

If G is flat, étale, ..., it suffices to consider flat, étale, ... T-schemes T'.
We show that under not very restrictive assumptions this is a closed subfunctor and thus represented by a closed subscheme $Y^G \subseteq Y$.

Theorem 2.1 ([DG, II, §1, Thm. 3.6]). *Let Y be a G-scheme over a scheme S. Assume that $G = \mathrm{Spec}_S \mathscr{A}$ is affine over S with $\mathscr{A}$ locally free on S (e.g. G finite flat over S or $S = \mathrm{Spec}\, K$, K a field) and that Y is separated over S. Then the fixed point subfunctor $\underline{Y}^G$ is a closed subfunctor of $\underline{Y}$.*

Proof. We use the functors $\underline{\mathrm{Mor}}_S(G, Y) : (S\textit{-schemes})^\circ \to (\textit{sets})$ defined by $\underline{\mathrm{Mor}}_S(G, Y)(T) = \mathrm{Mor}_T(G_T, Y_T)$ and $\underline{\mathrm{Mor}}_S(G, Y)(T) \to \underline{\mathrm{Mor}}_S(G, Y)(T')$, $\alpha \mapsto \alpha_{T'}$ for $T' \to T$.
The functor $\underline{Y}^G$ is given as the fiber product in the cartesian square

$$\begin{array}{ccc} \underline{Y}^G & \longrightarrow & \underline{\mathrm{Mor}}_S(G, Y) \\ \downarrow & & \downarrow \\ \underline{Y} & \longrightarrow & \underline{\mathrm{Mor}}_S(G, Y) \times_S \underline{\mathrm{Mor}}_S(G, Y) \end{array}$$

where $\underline{Y} \to \underline{\mathrm{Mor}}_S(G,Y) \times_S \underline{\mathrm{Mor}}_S(G,Y)$ is given by $y \mapsto ((g \mapsto gy),(g \mapsto y))$ (more precisely, $y \in Y(T)$ defines an element $\varphi \in \underline{\mathrm{Mor}}_S(G,Y)(T) = \mathrm{Mor}_T(G_T, Y_T)$ by $g \mapsto gy_{T'}$ for $T' \to T$, $g \in G_T(T')$, write this as $(g \mapsto gy)$) and $\underline{\mathrm{Mor}}_S(G,Y) \to \underline{\mathrm{Mor}}_S(G,Y) \times_S \underline{\mathrm{Mor}}_S(G,Y)$ by $\alpha \mapsto (\alpha,\alpha)$. The right vertical morphism identifies with $\underline{\mathrm{Mor}}_S(id_G,\Delta) : \underline{\mathrm{Mor}}_S(G,Y) \to \underline{\mathrm{Mor}}_S(G, Y \times_S Y)$ where $\Delta : Y \to Y \times_S Y$ is the diagonal morphism and a closed embedding since Y is separated over S.
To show that $\underline{Y}^G \to \underline{Y}$ is a closed subfunctor, it suffices to show that $\underline{\mathrm{Mor}}_S(id_G,\Delta)$ is a closed embedding. This follows from:

Let $i : V \to W$ be a closed embedding of S-schemes and $U = \mathrm{Spec}_S\,\mathscr{A}$ be an affine S-scheme with $\mathscr{A}$ locally free on S. Then $\underline{\mathrm{Mor}}_S(U,V) \to \underline{\mathrm{Mor}}_S(U,W)$ induced by i is a closed subfunctor.

Proof: To show that this subfunctor is closed, one has to show that for any S-scheme R and any morphism $\mathrm{Mor}_S(\,\cdot\,,R) \to \underline{\mathrm{Mor}}_S(U,W)$ given by $id_R \mapsto (\alpha : U_R \to W_R)$ there is a closed subscheme $R' \subseteq R$ such that for $\beta \in \mathrm{Mor}_S(T,R)$: β factors through R' if and only if $\alpha_\beta : U_T \to W_T$ (base extension of α via β) factors through $V_T \subseteq W_T$.
Let $\psi : \mathscr{A}_R \to \mathscr{B}$ be the surjective homomorphism of $\mathcal{O}_R$-algebras corresponding to the closed embedding $\mathrm{Spec}_R\,\mathscr{B} = \alpha^{-1}(V_R) \to U_R = \mathrm{Spec}_R\,\mathscr{A}_R$. Then $\ker\psi = 0$ if and only if α factors through V_R.
The question is local on R and $\mathscr{A}_R$ is locally free, so we may assume $\mathscr{A} \cong \bigoplus_{j\in J} \mathcal{O}_R^{(j)}$ as an $\mathcal{O}_R$-module for some index set J. The closed subscheme $R' \subseteq R$, defined by the ideal sheaf generated by the $\ker(\mathcal{O}_R^{(j)} \to \mathscr{B})$, $j \in J$, has the desired properties. $\square$

The equivariant Quot functor can be considered as a subfunctor of the original Quot functor (canonical inclusion by forgetting the G-sheaf structures) and as such compared with the fixed point subfunctor.

Proposition 2.2. *The equivariant Quot functor coincides with the fixed point subfunctor.*

Proof. A T-valued point of $\underline{\mathrm{Quot}}_{\mathscr{F}/X/S}$ given by a subsheaf $\mathscr{H} \subseteq \mathscr{F}_T$ is a T-valued point of the fixed point subfunctor if and only if for all flat T-schemes T' and $g \in G(T')$ the isomorphism $g_*\lambda_g^{\mathscr{F}_{T'}} : \mathscr{F}_{T'} = g_*g^*\mathscr{F}_{T'} \to g_*\mathscr{F}_{T'}$ given by the G-sheaf structure of $\mathscr{F}$ restricts to an isomorphism $\mathscr{H}_{T'} \to g_*\mathscr{H}_{T'}$. By remark 1.10 this is equivalent to the statement that $\mathscr{H}$ is a G-subsheaf of $\mathscr{F}_T$ or equivalently that it defines a T-valued point of the equivariant Quot functor. $\square$

Corollary 2.3. *Let S be a noetherian scheme, G be an affine group scheme over S such that $G = \mathrm{Spec}_S\,\mathscr{A}$ with $\mathscr{A}$ locally free on S. Let X be a (quasi)projective G-scheme over S and $\mathscr{F}$ be a coherent G-sheaf on X. Then the equivariant Quot functor $\underline{\mathrm{Quot}}^{G,p}_{\mathscr{F}/X/S}$ for a fixed Hilbert polynomial $p \in \mathbb{Q}[z]$ is represented by a (quasi)projective S-scheme $\mathrm{Quot}^{G,p}_{\mathscr{F}/X/S}$.*

Proof. Theorem 1.1 resp. its extension to the quasiprojective case, theorem 2.1, prop. 2.2. $\square$

2.2 The G-Hilbert scheme

In this subsection let S be a noetherian scheme over a field K. Let $f : X \to S$ be a (quasi) projective S-scheme and G be a group scheme over K. Assume that $G = \mathrm{Spec}\,A$ is affine with A cosemisimple. Let G operate on X over S, i.e. $G \times_K X \to X$ is an S-morphism. Let $\mathscr{F}$ be a coherent G-sheaf on X.

Remark 2.4. Here we have different base schemes for X and G. The results of the last subsection concerning representability are applicable, we may consider the S-morphism $G \times_K X \to X$ as an operation $G_S \times_S X \to X$ of the group scheme G_S and an equivariant sheaf on X either as a G-sheaf or as a G_S-sheaf. We write $\underline{\mathrm{Quot}}^{G/K}_{\mathscr{F}/X/S}$, $\underline{\mathrm{Quot}}^{G/K,p}_{\mathscr{F}/X/S}$ for the corresponding Quot functors.

In the equivariant case the Quot functors for a constant Hilbert polynomial $n \in \mathbb{N}$ have a finer decomposition given by the isomorphism classes of n-dimensional representations of G over K. For $n \in \mathbb{N}$ and V an isomorphism class of n-dimensional representations of G over K one has a subfunctor $\underline{\mathrm{Quot}}^{G/K,V}_{\mathscr{F}/X/S} \subseteq \underline{\mathrm{Quot}}^{G/K,n}_{\mathscr{F}/X/S}$ given by (T a locally noetherian S-scheme)

$$\underline{\mathrm{Quot}}^{G/K,V}_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T, \ \operatorname{supp}\mathscr{G} \textit{ finite over } T, \\ \textit{for } t \in T : H^0(X_t, \mathscr{G}_t) \cong V_{\kappa(t)} \textit{ as representations over } \kappa(t) \end{array} \right\}$$

where we have replaced "supp$\mathscr{G}$ *proper over* T, supp$\mathscr{G}_t$ 0-*dimensional*" by the equivalent condition "supp$\mathscr{G}$ *finite over* T" (here proper, quasifinite implies finite by [EGA, III (1), (4.4.2)] or [EGA, IV (3), (8.11.1)]).

Remark 2.5. As in the case of Hilbert schemes of n points in subsection 1.1 $f_{T*}\mathscr{G}$ is a locally free sheaf on T and the canonical homomorphisms $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} \to H^0(X_t, \mathscr{G}_t)$ of representations over $\kappa(t)$ (that can be constructed using the adjunction between inverse and direct image functors for G-sheaves) are isomorphisms. Thus one may rewrite the above functor as

$$\underline{\mathrm{Quot}}^{G/K,V}_{\mathscr{F}/X/S}(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } X_T, \\ \mathscr{G} \textit{ flat over } T, \ \operatorname{supp}\mathscr{G} \textit{ finite over } T, \\ \textit{for } t \in T : \kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} \cong V_{\kappa(t)} \textit{ as representations over } \kappa(t) \end{array} \right\}$$

Proposition 2.6. *For V an isomorphism class of n-dimensional representations of G over K the functor $\underline{\mathrm{Quot}}^{G/K,V}_{\mathscr{F}/X/S}$ is an open and closed subfunctor of $\underline{\mathrm{Quot}}^{G/K,n}_{\mathscr{F}/X/S}$.*

Proof. Let T be an S-scheme and $[\mathscr{F}_T \to \mathscr{G}]$ a quotient of G-sheaves defining a T-valued point of $\underline{\mathrm{Quot}}^{G/K,n}_{\mathscr{F}/X/S}$. The sheaf $f_{T*}\mathscr{G}$ is a locally free G-sheaf on T, as explained after proposition 1.13 it has a decomposition into isotypical components $f_{T*}\mathscr{G} = \bigoplus_i \mathscr{G}_i$ with $\mathscr{G}_i$ the components corresponding to the isomorphism classes V_i of simple representations of G over K.
The decomposition $f_{T*}\mathscr{G} = \bigoplus_i \mathscr{G}_i$ determines the decomposition $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} = \bigoplus_i (\kappa(t) \otimes_{\mathcal{O}_T} \mathscr{G}_i)$ into isotypical components, in particular the multiplicity of $(V_i)_{\kappa(t)}$ in $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G}$ is determined by the rank of $\mathscr{G}_i$, which is locally free as direct summand of the locally free sheaf $f_{T*}\mathscr{G}$. It follows, that these multiplicities are locally constant on T and therefore the condition $\kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathscr{G} \cong V_{\kappa(t)}$ is open and closed. □

Corollary 2.7. *If X is (quasi)projective over S, then the functor $\underline{\mathrm{Quot}}^{G/K,V}_{\mathscr{F}/X/S}$ is represented by a (quasi)projective S-scheme $\mathrm{Quot}^{G/K,V}_{\mathscr{F}/X/S}$.* □

For G finite over K of degree $|G|$ the G-Quot functor $\text{G-Quot}_{\mathscr{F}/X/S}$ arises by taking for V the regular representation of G. In particular, the G-Hilbert functor is defined by

$$\underline{\text{G-Hilb}}_S\, X(T) := \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{X_T} \to \mathcal{O}_Z \to 0] \textit{ on } X_T, \\ Z \textit{ finite flat over } T, \ \textit{for } t \in T : H^0(Z_t, \mathcal{O}_{Z_t}) \textit{ isomorphic} \\ \textit{to the regular representation} \end{array} \right\} \tag{2}$$

To summarize the arguments in this special case, $\underline{\mathrm{Hilb}}^{|G|}_S X$ is representable by a (quasi)projective S-scheme by theorem 1.1, $\underline{\text{G-Hilb}}_S\, X$ is an open and closed subfunctor of the equivariant Hilbert functor $\underline{\mathrm{Hilb}}^{G/K,|G|}_S X$ or, what amounts to the same by proposition 2.2, the fixed point subfunctor of $\underline{\mathrm{Hilb}}^{|G|}_S X$, which is a closed subfunctor of $\underline{\mathrm{Hilb}}^{|G|}_S X$ by theorem 2.1. Thus one has the result:

Corollary 2.8. *If X is (quasi)projective over S, then the functor $\underline{\text{G-Hilb}}_S\, X$ is represented by a (quasi)projective S-scheme $\text{G-Hilb}_S\, X$.* □

3 G-Hilb X and X/G

In this section about the relation between the G-Hilbert scheme and the quotient we will use some elementary facts about quotients of schemes by group schemes, notions such as categorical and geometric quotients, etc. as developed in [Mu, GIT], [Mu, AV], here for affine group schemes a geometric quotient will be assumed to be an affine morphism with the usual conditions (i)-(iv) of [Mu, GIT, Ch. 0, §2, Def. 6, p. 4]. In particular we will consider quotients of K-schemes X by finite group schemes $G = \operatorname{Spec} A$ over K with A cosemisimple. Then, in the affine case a geometric quotient $X = \operatorname{Spec} B \to X/G$ is constructed by taking invariants, that is $X/G = \operatorname{Spec} B^G$, and this can be applied to the general case, if X can be covered by G-stable affine open subschemes. If an affine geometric quotient $X \to X/G$ exists, then if X is algebraic (that is of finite type over K), X is finite over X/G and X/G is algebraic as well.

In the following let K be a field and $G = \operatorname{Spec} A$ a finite group scheme over K with A cosemisimple. Let S be a K-scheme and X an S-scheme with G-operation over S. The G-Hilbert functor for X is defined as in equation (2).

3.1 The morphism G-Hilb $X \to X/G$

For an extension field L of K an L-valued point of $\text{G-Hilb}_K X$, that is a finite closed subscheme $Z \subseteq X_L$ such that $H^0(Z, \mathcal{O}_Z)$ is isomorphic to the regular representation of G over L, is sometimes called a G-cluster. Since the regular representation only contains one copy of the trivial representation, the support of a G-cluster consists of only one G-orbit and thus defines a point of the quotient. We look for a morphism which contains this map as the map of points.

Lemma 3.1. *Let T be an S-scheme and $Z \subseteq X_T$ a closed subscheme defining an element of $\underline{\text{G-Hilb}}_S X(T)$. Then the projection $Z \to T$ is a geometric quotient of Z by G.* □

Theorem 3.2. *Let Y be a G-scheme over S with trivial G-operation and $\varphi : X \to Y$ an equivariant morphism. Then there is a unique morphism of functors (S-schemes)$^\circ \to$ (sets)*

$$\tau : \underline{\text{G-Hilb}}_S X \to \underline{Y}$$

such that for S-schemes T and $Z \in \underline{\text{G-Hilb}}_S X(T)$ with image $\tau(Z) \in Y(T)$ the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "\tau(Z)"] & Y \end{tikzcd} \tag{3}$$

commutes.

Proof. By Lemma 3.1 the morphism $Z \to T$ for $Z \in \underline{\text{G-Hilb}}_S X(T)$ is a geometric and thus also categorical quotient of Z by G. The existence and uniqueness of a morphism $\tau(Z)$ such that diagram (3) commutes follow from its universal property.
The maps $\underline{\text{G-Hilb}}_S X(T) \to Y(T)$, $Z \mapsto \tau(Z)$ are functorial in T, that is $\tau(Z_{T'}) = \tau(Z) \circ \alpha$ for morphisms $\alpha : T' \to T$ of S-schemes (by uniqueness of $\tau(Z_{T'})$). Therefore they define a morphism of functors $\tau : \underline{\text{G-Hilb}}_S X \to \underline{Y}$. □

Corollary 3.3. *There is a unique morphism of functors (S-schemes)$^\circ \to$ (sets)*

$$\tau : \underline{\text{G-Hilb}}_S X \to \underline{X/G}$$

such that for S-schemes T and $Z \in \underline{\text{G-Hilb}}_S\, X(T)$ with image $\tau(Z) \in X/G\,(T)$ the diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ T \arrow[r, "\tau(Z)"] & X/G \end{tikzcd} \tag{4}$$

commutes. □

Construction 3.4. We construct the morphism of the theorem in a more concrete way without using Lemma 3.1 by defining a morphism of S-functors $\underline{\text{G-Hilb}}_S\, X \to \underline{\text{Hilb}}^1_S\, Y \to \underline{Y}$.

Let $f : X \to S$, $g : Y \to S$ be S-schemes, $Y \to S$ separated. Let G operate on X over S and $\varphi : X \to Y$ be an equivariant morphism, Y with trivial G-operation. Similar to G-Hilbert functors for S-schemes without finiteness condition we define the functor $\underline{\text{Hilb}}^1_S\, Y$ by

$$\underline{\text{Hilb}}^1_S\, Y(T) = \left\{ \begin{array}{l} \textit{Quotient sheaves } [0 \to \mathscr{I} \to \mathcal{O}_{Y_T} \to \mathcal{O}_Z \to 0] \textit{ on } Y_T, \\ Z \textit{ finite flat over } T, \textit{ for } t \in T : h^0(Z_t, \mathcal{O}_{Z_t}) = 1 \end{array} \right\}$$

Let T be an S-scheme and $[\mathcal{O}_{X_T} \to \mathcal{O}_Z] \in \underline{\text{G-Hilb}}_S\, X(T)$. The restriction $\varphi_Z := (\varphi_T)|_Z : Z \to Y_T$ is a finite morphism. Because the G-operation on Y_T is trivial, the corresponding G-equivariant homomorphism $\mathcal{O}_{Y_T} \to \varphi_{Z*}\mathcal{O}_Z$ factors through $\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G$ (this corresponds to the universal property of the quotient $Z \to T$ used in the proof of the theorem).

The homomorphism $\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G$ is surjective because already $g_{T*}O_{Y_T} \to g_{T*}(\varphi_{Z*}\mathcal{O}_Z)^G$ is (the composition of the canonical homomorphisms $\mathcal{O}_T \to g_{T*}O_{Y_T} \to g_{T*}(\varphi_{Z*}\mathcal{O}_Z)^G = (f_{T*}\mathcal{O}_Z)^G$ is an isomorphism). Since $\mathcal{O}_Z$ is flat over T, $\varphi_{Z*}\mathcal{O}_Z$ is flat over T and the direct summand $(\varphi_{Z*}\mathcal{O}_Z)^G$ is flat as well. The representation $H^0(Y_t, (\varphi_{Z*}\mathcal{O}_Z)_t) \cong \kappa(t) \otimes_{\mathcal{O}_T} f_{T*}\mathcal{O}_Z \cong H^0(X_t, (\mathcal{O}_Z)_t)$ over $\kappa(t)$ (see also remark 2.5) is isomorphic to the regular representation, therefore $h^0(Y_t, (\varphi_{Z*}\mathcal{O}_Z)^G_t) = 1$ for $t \in T$. Thus the quotient $[\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G]$ defines an element of $\underline{\text{Hilb}}^1_S\, Y(T)$.

The maps $\underline{\text{G-Hilb}}_S\, X(T) \to \underline{\text{Hilb}}^1_S\, Y(T)$, $[\mathcal{O}_{X_T} \to \mathcal{O}_Z] \mapsto [\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G]$ are functorial in T, i.e. for S-morphisms $T' \to T$ the quotients $[\mathcal{O}_{Y_{T'}} \to (\varphi_{Z*}\mathcal{O}_Z)^G_{T'}]$ and $[\mathcal{O}_{Y_{T'}} \to (\varphi_{Z_{T'}*}\mathcal{O}_{Z_{T'}})^G]$ coincide: The natural homomorphisms of G-equivariant $\mathcal{O}_{Y_{T'}}$-algebras $(\varphi_{Z*}\mathcal{O}_Z)_{T'} \to \varphi_{Z_{T'}*}\mathcal{O}_{Z_{T'}}$ arising from the diagrams

$$\begin{tikzcd} Z_{T'} \arrow[r] \arrow[d, "\varphi_{Z_{T'}}"'] & Z \arrow[d, "\varphi_Z"] \\ Y_{T'} \arrow[r] & Y_T \end{tikzcd}$$

are isomorphisms (φ_Z is finite). So these maps define a morphism $\underline{\text{G-Hilb}}_S\, X \to \underline{\text{Hilb}}^1_S\, Y$.

Taking into account the canonical morphism $\underline{\text{Hilb}}^1_S\, Y \to \underline{Y}$ that arises exactly as in proposition 1.3, one obtains a morphism $\underline{\text{G-Hilb}}_S\, X \to \underline{Y}$ such that the diagrams (3) commute: The element $[\mathcal{O}_{Y_T} \to (\varphi_{Z*}\mathcal{O}_Z)^G] \in \underline{\text{Hilb}}^1_S\, Y(T)$ corresponds to a closed subscheme $W \subseteq Y_T$ over which $\varphi_Z : Z \to Y_T$ factors. $\tau(Z)$ then is constructed using the inverse of the isomorphism $W \to T$:

$$\begin{tikzcd} Z \arrow[d, "\varphi_Z"'] & \subseteq X_T \arrow[r] \arrow[d, "\varphi_T"] & X \arrow[d, "\varphi"] \\ W \arrow[dr, phantom] & \subseteq Y_T \arrow[r] \arrow[d] & Y \\ & T \arrow[ul, dashed, "\wr"] \arrow[ur, dashed, "\tau(Z)"'] & \end{tikzcd}$$

Remark 3.5. A similar construction can be used in more general situations to form morphisms $\underline{\text{G-Hilb}}_S\, X \to \underline{\text{H-Hilb}}_S Y$ or at least morphisms defined on open subschemes.

3.2 Change of the base scheme of G-Hilbert functors

In this subsection we pursue the general idea of varying the base scheme for G-Hilbert functors. The base scheme always has trivial G-operation, a in some sense maximal possible base scheme, that is one with minimal fibers, for the G-Hilbert functor of a given G-scheme X is the quotient X/G. We show that $\underline{\text{G-Hilb}}_S\, X$ considered as a X/G-functor via the morphism τ of corollary 3.3 is isomorphic to $\underline{\text{G-Hilb}}_{X/G}\, X$.

We begin with some generalities on changing the base scheme of functors with respect to a morphism $\varphi : S' \to S$.

- base restriction: For a functor $F' : (S'\text{-}schemes)^\circ \to (sets)$ define ${}_SF' : (S\text{-}schemes)^\circ \to (sets)$ by taking disjoint unions

$$ {}_SF'(T) := \bigsqcup \big\{ F'(\alpha) \;\big|\; \alpha \in \operatorname{Mor}_S(T, S') \big\} $$

For an S'-scheme Y the corresponding construction is to consider Y as an S-scheme by composing the structure morphism $Y \to S'$ with $S' \to S$.

- base extension: For a functor $F : (S\text{-}schemes)^\circ \to (sets)$ define $F_{S'} : (S'\text{-}schemes)^\circ \to (sets)$ by

$$ F_{S'}(\alpha : T \to S') = F(\varphi \circ \alpha) $$

which is the restriction of F to the category of S'-schemes. $F_{S'}$ also can be realized as the fibered product $F \times_{\underline{S}} \underline{S'}$ considered as S'-functor. For an S-scheme X this corresponds to the usual base extension $X_{S'} = X \times_S S'$.

- If in addition a morphism of S-functors $\psi : F \to \underline{S'}$ is given, then one can consider F as a functor on the category of S'-schemes via ψ, in other words let $F_{(S',\psi)} : (S'\text{-}schemes)^\circ \to (sets)$ be given by

$$ F_{(S',\psi)}(\alpha : T \to S') = \{\beta \in F(\varphi \circ \alpha) \mid \psi(\beta) = \alpha\} = \{\beta \in F_{S'}(\alpha) \mid \psi(\beta) = \alpha\} $$

For schemes this means to consider an S-scheme X as an S'-scheme via a given S-morphism $X \to S'$. Note that $F_{(S',\psi)}$ can be considered as subfunctor of $F_{S'}$ and that ${}_S(F_{(S',\psi)}) \cong F$.

Remark 3.6. (Base extension for G-Hilbert schemes).
(1) Let X be a G-scheme over S and S' an S-scheme. Then there is the isomorphism of S'-functors

$$ (\underline{\text{G-Hilb}}_S\, X)_{S'} \cong \underline{\text{G-Hilb}}_{S'}\, X_{S'} $$

which derives from the natural isomorphisms $X \times_S T \cong X_{S'} \times_{S'} T$ for S'-schemes T.
(2) Sometimes, preferably in the case $S' = \operatorname{Spec} K' \to S = \operatorname{Spec} K$ with K, K' fields, it can be useful to involve the group scheme into the base change. Then

$$ (\underline{\text{G-Hilb}}_K\, X)_{K'} \cong \underline{\text{G-Hilb}}_{K'}\, X_{K'} \cong \underline{\text{G}_{K'}\text{- Hilb}}_{K'} X_{K'} $$

Theorem 3.7. *Let Y be a G-scheme over S with trivial G-operation and $\varphi : X \to Y$ an equivariant morphism. Let $\tau : \underline{\text{G-Hilb}}_S\, X \to \underline{Y}$ be the morphism constructed in theorem 3.2. Then there is an isomorphism of functors $(Y\text{-}schemes)^\circ \to (sets)$*

$$ (\underline{\text{G-Hilb}}_S\, X)_{(Y,\tau)} \cong \underline{\text{G-Hilb}}_Y\, X $$

Proof. Both functors are subfunctors of $(\underline{\text{G-Hilb}}_S\,X)_Y : (Y\textit{-schemes})^\circ \to (\textit{sets})$:

$$(\underline{\text{G-Hilb}}_S\,X)_{(Y,\tau)}(\alpha : T \to Y) = \{Z \in (\underline{\text{G-Hilb}}_S\,X)_Y(\alpha : T \to Y) \mid \tau(Z) = \alpha\}$$
$$\underline{\text{G-Hilb}}_Y\,X(T) = \{Z \in (\underline{\text{G-Hilb}}_S\,X)_Y(T) \mid Z \hookrightarrow X \times_S T \textit{ factors through } X \times_Y T\}$$

We show that they coincide. Let T be a Y-scheme and $Z \subseteq X \times_S T$ a closed subscheme defining an element of $(\text{G-Hilb}_S\,X)_Y(T)$. Then there are the equivalences

$$\begin{aligned} Z \in \underline{\text{G-Hilb}}_Y\,X(T) &\iff Z \hookrightarrow X \times_S T \textit{ factors through } X \times_Y T \\ &\iff \textit{diagram (3) commutes for } Z \textit{ and the given morphism } T \to Y \\ &\iff Z \in (\underline{\text{G-Hilb}}_S\,X)_{(Y,\tau)}(T) \end{aligned}$$

where for the last one one uses uniqueness of a morphism $T \to Y$ making diagram (3) commute. $\square$

Corollary 3.8. *Let $\tau : \underline{\text{G-Hilb}}_S\,X \to \underline{X/G}$ be the morphism of corollary 3.3. Then there is an isomorphism of functors $(X/G\textit{-schemes})^\circ \to (\textit{sets})$*

$$(\underline{\text{G-Hilb}}_S\,X)_{(X/G,\tau)} \cong \underline{\text{G-Hilb}}_{X/G}\,X$$

$\square$

Remark 3.9. We explicitly write down some direct consequences.
(1) The fact that there is an isomorphism $(\underline{\text{G-Hilb}}_S\,X)_{(X/G,\tau)} \cong \underline{\text{G-Hilb}}_{X/G}\,X$ of functors $(X/G\textit{-schemes})^\circ \to (\textit{sets})$ of course implies an isomorphism $\underline{\text{G-Hilb}}_S\,X \cong {}_S(\underline{\text{G-Hilb}}_{X/G}\,X)$ of functors $(S\textit{-schemes})^\circ \to (\textit{sets})$.
(2) If $\underline{\text{G-Hilb}}_{X/G}\,X$ is representable, then so is $\underline{\text{G-Hilb}}_S\,X$ and in this case the corresponding schemes $\text{G-Hilb}_{X/G}\,X$ and $\text{G-Hilb}_S\,X$ are isomorphic as X/G-schemes, i.e. there is the commutative diagram

$$\begin{tikzcd} \text{G-Hilb}_{X/G}\,X \arrow[rr, leftrightarrow, "\sim"] \arrow[dr] & & \text{G-Hilb}_S\,X \arrow[dl, "\tau"] \\ & X/G & \end{tikzcd}$$

(3) In the situation of (2), after base restriction by $X/G \to S$ there is the isomorphism of S-schemes ${}_S(\text{G-Hilb}_{X/G}\,X) \cong \text{G-Hilb}_S\,X$.

Remark 3.10. (Fibers of $\tau : \text{G-Hilb}_S\,X \to X/G$).
Assume that $\underline{\text{G-Hilb}}_{X/G}\,X$ is representable (then so is $\underline{\text{G-Hilb}}_S\,X$). For points $y \in X/G$ one may consider the fibers of the morphism $\tau : \text{G-Hilb}_S\,X \to X/G$ of corollary 3.3. Corollary 3.8 gives an isomorphism $\text{G-Hilb}_S\,X \cong {}_S(\text{G-Hilb}_{X/G}\,X)$ which identifies τ with the structure morphism of $\text{G-Hilb}_{X/G}\,X$. So the fibers of τ are the fibers $(\text{G-Hilb}_{X/G}\,X)_y$ of the X/G-scheme $\text{G-Hilb}_{X/G}\,X$, these are isomorphic to G-Hilbert schemes $\text{G-Hilb}_{\kappa(y)}\,X_y$ over $\kappa(y)$ of the G-schemes X_y which are the fibers of $X \to X/G$ over y.

Another application (result also contained in [Té04]):

Corollary 3.11. *Let X, Y be G-schemes over S, Y with trivial G-operation. Then*

$$\underline{\text{G-Hilb}}_S\,X \times_S Y \cong (\underline{\text{G-Hilb}}_S\,X) \times_{\underline{S}} \underline{Y}$$

Proof. The projection $X \times_S Y \to Y$ is G-equivariant, theorem 3.2 then constructs a morphism of S-functors $\tau : \underline{\text{G-Hilb}}_S\,X \times_S Y \to \underline{Y}$, by theorem 3.7 there is an isomorphism of Y-functors $(\underline{\text{G-Hilb}}_S\,X \times_S Y)_{(Y,\tau)} \cong \underline{\text{G-Hilb}}_Y\,X \times_S Y$ and $\underline{\text{G-Hilb}}_Y\,X \times_S Y \cong (\underline{\text{G-Hilb}}_S\,X) \times_{\underline{S}} \underline{Y}$ by the usual base extension. Restricting the base to S yields the result. $\square$

3.3 G-Hilbert schemes and G-Grassmannians

In this subsection we show in the case that the G-scheme X is affine over S, that there is a natural closed embedding of the G-Hilbert functor $\underline{\text{G-Hilb}}_S\, X$ into a G-Grassmannian functor, this implies that $\underline{\text{G-Hilb}}_S\, X$ is representable. This construction is useful for X finite over S, in which case $\text{G-Hilb}_S\, X$ is projective over S.

A special case of a G-Quot functor is the G-Grassmannian functor $\underline{\text{G-Grass}}_S(\mathscr{F}) = \underline{\text{G-Quot}}_{\mathscr{F}/S/S}$ for a quasicoherent G-sheaf $\mathscr{F}$ on S:

$$\underline{\text{G-Grass}}_S(\mathscr{F})(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0] \textit{ on } T, \ \mathscr{G} \textit{ locally free} \\ \textit{of finite rank with fibers isomorphic to the regular representation} \end{array} \right\}$$

Let X be a G-scheme over S, assume that X is affine over S, $X = \operatorname{Spec}_S \mathscr{B}$ for a quasicoherent G-sheaf of $\mathcal{O}_S$-algebras. One may rewrite the G-Hilbert functor for $X \to S$ as

$$\underline{\text{G-Hilb}}_S\, X(T) = \left\{ \begin{array}{l} \textit{Quotient } G\textit{-sheaves } [0 \to \mathscr{I} \to \mathscr{B}_T \to \mathscr{C} \to 0] \\ \textit{of } \mathscr{B}_T\textit{-modules on } T, \ \mathscr{C} \textit{ locally free of finite rank} \\ \textit{with fibers isomorphic to the regular representation} \end{array} \right\}$$

There is a morphism of S-functors $\underline{\text{G-Hilb}}_S\, X \to \underline{\text{G-Grass}}_S(\mathscr{B})$ consisting of injective maps

$$\underline{\text{G-Hilb}}_S\, X(T) \to \underline{\text{G-Grass}}_S(\mathscr{B})(T), \quad [\mathscr{B}_T \to \mathscr{C}] \mapsto [\mathscr{B}_T \to \mathscr{C}]$$

defined by forgetting the algebra structure of $\mathscr{B}$, this way $\underline{\text{G-Hilb}}_S\, X$ becomes a subfunctor of $\underline{\text{G-Grass}}_S(\mathscr{B})$. The essential point we will show is the closedness of the additional condition for $\mathscr{I} \subseteq \mathscr{B}_T$ to be an ideal of $\mathscr{B}_T$, consequence will be the following theorem.

Theorem 3.12. *Let $X = \operatorname{Spec}_S \mathscr{B} \to S$ be an affine S-scheme with G-operation over S. Then the natural morphism of S-functors*

$$\underline{\text{G-Hilb}}_S\, X \to \underline{\text{G-Grass}}_S(\mathscr{B})$$

is a closed embedding.

Proof. We show that the canonical inclusion defined above is a closed embedding.

One has to show that for any S-scheme S' and any S'-valued point of the Grassmannian $[0 \to \mathscr{H} \xrightarrow{\varphi} \mathscr{B}_{S'} \to \mathscr{G} \to 0]$ there is a closed subscheme $Z \subseteq S'$ such that for every S-morphism $\alpha : T \to S'$: α factors through Z if and only if the quotient $[\mathscr{B}_T \to \alpha^*\mathscr{G}] \in \underline{\text{G-Grass}}_S(\mathscr{B})(T)$ determined by α comes from a quotient of the Hilbert functor.

For $\alpha : T \to S'$ the quotient $[\alpha^*\mathscr{H} \xrightarrow{\alpha^*\varphi} \mathscr{B}_T \to \alpha^*\mathscr{G} \to 0]$ of $\mathcal{O}_T$-modules comes from a quotient of $\mathscr{B}_T$-modules if and only if the image of $\alpha^*\mathscr{H}$ in $\mathscr{B}_T$ is a $\mathscr{B}_T$-submodule. Equivalently, the composition $\chi_T : \mathscr{B}_T \otimes_{\mathcal{O}_T} \alpha^*\mathscr{H} \to \mathscr{B}_T \otimes_{\mathcal{O}_T} (\alpha^*\varphi)(\alpha^*\mathscr{H}) \to \mathscr{B}_T \to \alpha^*\mathscr{G}$, where the arrow in the middle is defined using the multiplication map $\mathscr{B}_T \otimes_{\mathcal{O}_T} \mathscr{B}_T \to \mathscr{B}_T$ of the algebra $\mathscr{B}_T$, is zero.

To define Z, consider $\chi_{S'} : \mathscr{B}_{S'} \otimes_{\mathcal{O}_{S'}} \mathscr{H} \to \mathscr{G}$. Define Z to be the closed subscheme with ideal sheaf $\mathscr{I}$ that is the ideal of $\mathcal{O}_{S'}$ minimal with the property $\operatorname{im}(\chi_{S'}) \subseteq \mathscr{I}\mathscr{G}$. Locally $\mathscr{I}$ can be described as follows: If $\mathscr{G}|_U \cong \bigoplus_j \mathcal{O}_U^{(j)}$ for an open $U \subseteq S'$, then $\mathscr{I}$ is the ideal sheaf generated by the images of the coordinate maps $\chi_U^{(j)} : (\mathscr{B}_{S'} \otimes_{\mathcal{O}_{S'}} \mathscr{H})|_U \to \mathcal{O}_U^{(j)}$.

This Z has the required property: One has to show that $\alpha : T \to S'$ factors through Z if and only if $(\alpha^*\varphi)(\alpha^*\mathscr{H}) \subseteq \mathscr{B}_T$ is a $\mathscr{B}_T$-submodule. χ_T identifies with $\alpha^*\chi_{S'}$. The question is local

on S', assume that $\mathscr{G} \cong \bigoplus_j \mathcal{O}_{S'}^{(j)}$. Then there are the equivalences:

$$\begin{aligned}
(\alpha^*\varphi)(\alpha^*\mathscr{H}) \textit{ is a } \mathscr{B}_T\textit{-submodule} &\iff \operatorname{im}(\chi_T) = 0 \textit{ in } \alpha^*\mathscr{G} \\
&\iff \operatorname{im}(\alpha^*\chi_{S'}) = 0 \textit{ in } \alpha^*\mathscr{G} \\
&\iff \forall\, j:\ \operatorname{im}(\alpha^*\chi_{S'}^{(j)}) = 0 \textit{ in } \mathcal{O}_T^{(j)} \\
&\iff \forall\, j:\ \operatorname{im}(\chi_{S'}^{(j)}) \subseteq \ker(\mathcal{O}_{S'} \to \alpha_*\mathcal{O}_T) \\
&\iff \mathscr{I} \subseteq \ker(\mathcal{O}_{S'} \to \alpha_*\mathcal{O}_T) \\
&\iff \alpha : T \to S \textit{ factors through } Z
\end{aligned}$$

□

The Grassmannian functor $\underline{\mathrm{Grass}}_S^n(\mathscr{B})$ is representable (theorem 1.2) and so is the G-Grassmannian as a component of the equivariant Grassmannian $\underline{\mathrm{Grass}}_S^{G,n}(\mathscr{B})$ (same argument as in proposition 2.6), which is a closed subfunctor of $\underline{\mathrm{Grass}}_S^n(\mathscr{B})$ (proposition 2.2 and theorem 2.1). Further, if $f : X \to S$ is finite, then $\mathscr{B} = f_*\mathcal{O}_X$ is quasicoherent of finite type, the Grassmannian $\mathrm{Grass}_S^n(\mathscr{B})$ and thereby the G-Grassmannian $\text{G-Grass}_S(\mathscr{B})$ projective over S.

Corollary 3.13. *If $X \to S$ is finite, then the S-functor $\underline{\text{G-Hilb}}_S\, X$ is representable by a projective S-scheme.* □

3.4 Representability of G-Hilbert functors

For finite group schemes an affine quotient morphism of an algebraic K-scheme $X \to X/G$ is finite. This allows to use the closed embedding of $\underline{\text{G-Hilb}}_{X/G}\, X$ into a Grassmannian of theorem 3.12 to prove that the G-Hilbert functor $\underline{\text{G-Hilb}}_{X/G}\, X$ is represented by a scheme $\text{G-Hilb}_{X/G}\, X$ projective over X/G directly without using theorem 1.1 about representability of Quot and Hilbert functors. The behaviour of G-Hilbert functors under variation of the base scheme then implies that $\underline{\text{G-Hilb}}_K\, X$ is representable and the morphism $\tau : \text{G-Hilb}_K\, X \to X/G$ projective.

Theorem 3.14. *(Representability of $\underline{\text{G-Hilb}}_K\, X$).*
Let $G = \operatorname{Spec} A$ be a finite group scheme over a field K with A cosemisimple. Let X be a G-scheme algebraic over K and assume that a geometric quotient $\pi : X \to X/G$, π affine, of X by G exists. Then the G-Hilbert functor $\underline{\text{G-Hilb}}_K\, X$ is represented by an algebraic K-scheme $\text{G-Hilb}_K\, X$ and the morphism $\tau : \text{G-Hilb}_K\, X \to X/G$ of corollary 3.3 is projective.

Proof. Since the group scheme G is finite and X is algebraic, π is a finite morphism and X/G is algebraic. Thus corollary 3.13 applies to the G-Hilbert functor $\underline{\text{G-Hilb}}_{X/G}\, X$ and implies that $\underline{\text{G-Hilb}}_{X/G}\, X$ is represented by an algebraic K-scheme $\text{G-Hilb}_{X/G}\, X$ projective over X/G.

Corollary 3.3 constructs a morphism $\tau : \underline{\text{G-Hilb}}_K\, X \to \underline{X/G}$, by corollary 3.8 there is the isomorphism of X/G-functors $(\underline{\text{G-Hilb}}_K\, X)_{(X/G,\tau)} \cong \underline{\text{G-Hilb}}_{X/G}\, X$. In particular there is an isomorphism of K-functors $\underline{\text{G-Hilb}}_K\, X \cong {}_K(\underline{\text{G-Hilb}}_{X/G}\, X)$ that identifies τ with the structure morphism of $\underline{\text{G-Hilb}}_{X/G}\, X$.

It follows that $\underline{\text{G-Hilb}}_K\, X$ is represented by an algebraic K-scheme $\text{G-Hilb}_K\, X$ and that there is an isomorphism of K-schemes $\text{G-Hilb}_K\, X \cong \text{G-Hilb}_{X/G}\, X$ that identifies the morphism $\tau : \text{G-Hilb}_K\, X \to X/G$ with the structure morphism of $\text{G-Hilb}_{X/G}\, X$ which is projective. □

3.5 Free operation

Let the finite group scheme G over K operate on an algebraic K-scheme X.

Proposition 3.15. *Let $f : X \to S$ be a finite flat morphism and let G operate on X over S such that the fibers $f_*\mathcal{O}_X \otimes_{\mathcal{O}_S} \kappa(s)$ for $s \in S$ are isomorphic to the regular representation. Then there is an isomorphism of S-functors $\underline{\text{G-Hilb}}_S\, X \cong \underline{S}$.*

Proof. For any S-scheme T X_T is finite flat over T with fibers isomorphic to the regular representation and the only closed subscheme of X_T with this property. Thus there are canonical bijections $\underline{\text{G-Hilb}}_S\, X(T) \leftrightarrow S(T)$ functorial in T. $\square$

Corollary 3.16. *If $X \to S$ is a G-torsor, then $\underline{\text{G-Hilb}}_S\, X \cong \underline{S}$.* $\square$

For the following corollaries assume that a geometric quotient $\pi : X \to X/G$, π affine, exists. Then by theorem 3.14 $\underline{\text{G-Hilb}}_K\, X$ is representable by an algebraic K-scheme $\text{G-Hilb}_K\, X$ and there is the projective morphism $\tau : \text{G-Hilb}_K\, X \to X/G$.

Corollary 3.17. *If the operation of G on X is free, then the morphism $\tau : \text{G-Hilb}_K\, X \to X/G$ is an isomorphism.*

Proof. The quotient morphism $\pi : X \to X/G$ is a G-torsor since the operation is free [Mu, AV, Ch. III.12, Thm. 1, p. 111,112], [Mu, GIT, Ch. 0, §4, Prop. 9, p. 16]. Then by corollary 3.16 there is an isomorphism of X/G-schemes $\text{G-Hilb}_{X/G}\, X \cong X/G$ and therefore $\tau : \text{G-Hilb}_K\, X \to X/G$ an isomorphism. $\square$

Corollary 3.18. *Assume that X is an irreducible K-variety. Let G operate on X such that the operation is free on a dense open subscheme. Then there is a unique irreducible component W of $(\text{G-Hilb}_K\, X)_{\text{red}}$ such that $\tau|_W : W \to X/G$ is birational. In particular, if $\text{G-Hilb}_K\, X$ is reduced and irreducible then τ is birational.*

Proof. Let $U \subseteq X$ be a G-stable dense open subscheme on which the G-operation is free. Then U/G is open dense in X/G, the restriction $\tau|_{\tau^{-1}(U/G)} : \tau^{-1}(U/G) \to U/G$ is an isomorphism by corollary 3.17 and $W :=$ *closure of* $\tau^{-1}(U/G)$ *in* $\text{G-Hilb}_K\, X$ is the unique irreducible component of $(\text{G-Hilb}_K\, X)_{\text{red}}$ such that $\tau|_W : W \to X/G$ is birational. $\square$

Corollary 3.19. *Let G operate on X such that the operation is free on a dense open subscheme. Then, if the quotient morphism $\pi : X \to X/G$ is flat, $\tau : \text{G-Hilb}_K\, X \to X/G$ is an isomorphism.*

Proof. The operation is free on an open dense subscheme and there the representations on the fibers of π are isomorphic to the regular representation. The isomorphism class of the representations on the fibers is locally constant, since $\pi_*\mathcal{O}_X$ is locally free and so are the direct summands corresponding to the isomorphism classes of simple representations of G over K – their rank determines the multiplicity of the corresponding simple representation in the representation on the fibers (as in the proof of proposition 2.6). Therefore the representation $\pi_*\mathcal{O}_X \otimes_{\mathcal{O}_{X/G}} \kappa(y)$ is isomorphic to the regular representation over $\kappa(y)$ for any $y \in X/G$. Then by proposition 3.15 $\text{G-Hilb}_{X/G}\, X \cong X/G$ as X/G-schemes and thus $\tau : \text{G-Hilb}_K\, X \to X/G$ is an isomorphism. $\square$

4 Differential study of G-Hilbert schemes

4.1 The sheaf of relative differentials and relative tangent spaces

Since the additional conditions defining the G-Quot functor are open and closed, its differential study reduces to that of the equivariant Quot functor. Let $\underline{Q}$ be an open and closed component of an equivariant Quot functor $\underline{\mathrm{Quot}}^G_{\mathscr{F}/X/S}$, assume that $\underline{Q}$ is represented by an S-scheme Q.

For an S-morphism $q : T \to Q$ and a quasicoherent $\mathcal{O}_T$-module $\mathscr{M}$ one defines a sheaf of sets $\mathscr{E}$ on T that assembles the sets of local extensions of q to morphisms $\widetilde{q} : \widetilde{T} \to Q$, where $(\widetilde{T}, \mathcal{O}_{\widetilde{T}}) = (T, \mathcal{O}_T \oplus \varepsilon\mathscr{M})$, $\mathscr{M}$ considered as an ideal of square zero. The relations between differentials, derivations and infinitesimal extensions of morphisms are subject of [EGA, IV (4), (16.5)]. Then, by a result about flat deformations of equivariant quotient sheaves similar to [Gr61, Prop. 5.1] one obtains the theorem below. Note that for only this purpose it is not necessary to assume $\mathscr{F}$ flat over S. Further, in this special situation, where there is a morphism $\widetilde{T} \to T$ such that $T \to \widetilde{T} \to T$ is the identity, there is always a natural zero-deformation resp. zero-extension, the sheaf $\mathscr{E}$ has a natural $\mathcal{O}_T$-module structure and is isomorphic, not just a pseudo-torsor under $\mathscr{A}$.

Theorem 4.1. *Let Q be as above, T be an S-scheme, $\mathscr{M}$ a quasicoherent $\mathcal{O}_T$-module. Let $q : T \to Q$ be a morphism over S, $[0 \to \mathscr{H} \to \mathscr{F}_T \to \mathscr{G} \to 0]$ the corresponding quotient of G-sheaves on X_T. Then there is an isomorphism of $\mathcal{O}_T$-modules*

$$\mathscr{H}om_{\mathcal{O}_T}(q^*\Omega_{Q/S}, \mathscr{M}) \cong \mathscr{A}$$

where $\mathscr{A}$ is given by

$$\mathscr{A}(U) = \mathrm{Hom}^G_{X_U}(\mathscr{H}|_{X_U}, \mathscr{G}|_{X_U} \otimes_{\mathcal{O}_U} \mathscr{M}|_U))$$

for open $U \subseteq T$. □

We consider the case of tangent spaces, at the same time we specialize to G-Hilbert schemes. The following well known result describes tangent spaces of $\text{G-Hilb}_K X$:

Corollary 4.2. *(Tangent spaces of* $\text{G-Hilb}_K X$*).*
Let $h : \operatorname{Spec} L \to \text{G-Hilb}_K X$, L an extension field of K, be a morphism of K-schemes corresponding to a quotient $[0 \to \mathscr{I} \to \mathcal{O}_{X_L} \to \mathcal{O}_Z \to 0]$ of G-sheaves on X_L. Then one has

$$T_h \, \text{G-Hilb}_K X \cong \mathrm{Hom}^G_{X_L}(\mathscr{I}, \mathcal{O}_Z)$$

for the tangent space of $\text{G-Hilb}_K X$ *at h.* □

The theorem also applies to relative G-Hilbert schemes, either directly or by means of the preceding corollary applied to the fibers over points of X/G.

Corollary 4.3. *(Relative tangent spaces of* $\text{G-Hilb}_{X/G} X$ *over X/G).*
Let $h : \operatorname{Spec} L \to \text{G-Hilb}_{X/G} X$, L an extension field of K, be a morphism lying over a morphism of K-schemes $h_0 : \operatorname{Spec} L \to X/G$. Let $[0 \to \mathscr{I} \to \mathcal{O}_{X_{h_0}} \to \mathcal{O}_Z \to 0]$ be the quotient of G-sheaves corresponding to h, where $X_{h_0} := \operatorname{Spec} L \times_{X/G} X$, the fiber product formed by h_0. Then one has

$$T_h \, \text{G-Hilb}_{X/G} X \cong \mathrm{Hom}^G_{X_{h_0}}(\mathscr{I}, \mathcal{O}_Z)$$

for the relative tangent space of $\text{G-Hilb}_{X/G} X$ *at h.* □

Since the morphism $\tau : \text{G-Hilb}_K X \to X/G$ defined in corollary 3.3 identifies with the structure morphism of $\text{G-Hilb}_{X/G} X$ by corollary 3.8, corollary 4.3 describes as well relative tangent spaces $\text{G-Hilb}_K X$ over X/G with respect to τ.

Example 4.4. Let $X = \mathbb{A}^n_K$, $G = \operatorname{Spec} A \subset \mathrm{GL}(n, K)$ a finite subgroup scheme with A cosemisimple. Then a geometric quotient $\pi : \mathbb{A}^n_K \to \mathbb{A}^n_K/G$ exists, the G-Hilbert functor is represented by a K-scheme $\text{G-Hilb}_K \mathbb{A}^n_K$, there is the projective morphism $\tau : \text{G-Hilb}_K \mathbb{A}^n_K \to \mathbb{A}^n_K/G$. Let $S := K[x_1, \ldots, x_n]$, let $O \in \mathbb{A}^n_K$ be the origin, $\mathfrak{m} \subset S$ the corresponding maximal ideal, $\overline{O} := \pi(O) \in \mathbb{A}^n_K/G$ with corresponding maximal ideal $\mathfrak{n} \subset S^G$, let $\overline{S} := S/\mathfrak{n}S$. An L-valued point of the fiber $E := \tau^{-1}(\overline{O})$ corresponds to a G-cluster defined by an ideal $I \subset S_L$ such that $\mathfrak{n}_L \subseteq I$ or equivalently an ideal $\overline{I} \subset \overline{S}_L = S_L/\mathfrak{n}_L S_L$.
The tangent space of $\text{G-Hilb}_K \mathbb{A}^n_K$ (over K) at I is the L-vector space

$$T_I \,\text{G-Hilb}_K \mathbb{A}^n_K \cong \operatorname{Hom}^G_{S_L}(I, S_L/I)$$

The relative tangent space of $\text{G-Hilb}_K \mathbb{A}^n_K$ over $\mathbb{A}^n_K/G$ at I or equivalently the tangent space of the fiber $E \cong \text{G-Hilb}_K \pi^{-1}(\overline{O})$ over $\overline{O}$ at I is the L-vector space

$$T_I E \cong \operatorname{Hom}^G_{\overline{S}_L}(\overline{I}, \overline{S}_L/\overline{I})$$

4.2 Relative tangent spaces and stratification

In [ItNm96], [ItNm99] has been defined a certain stratification of the G-Hilbert scheme, that will at least partially provided with a geometric meaning in this subsection.

For simplicity let $K = \mathbb{C}$, consider the situation in example 4.4, that is $X = \mathbb{A}^n_{\mathbb{C}}$, $G \subset \mathrm{GL}(n, \mathbb{C})$ a finite subgroup. Using the notations introduced there, for an ideal $I \subset S$ with $\mathfrak{n} \subseteq I$ or equivalently $\overline{I} \subseteq \overline{S}$ defining a G-cluster and thus a $\mathbb{C}$-valued point of $E = \tau^{-1}(\overline{O})$ one can consider the representation

$$\overline{I}/\overline{\mathfrak{m}}\overline{I} \cong I/(\mathfrak{m}I + \mathfrak{n}S)$$

This representation has been used in [ItNm96], [ItNm99] in the case of finite subgroups $G \subset \mathrm{SL}(2, \mathbb{C})$ to give a natural construction for the bijection observed in [McK80] between isomorphism classes of nontrivial irreducible representations of G and irreducible components of the exceptional divisor of the minimal resolution $\text{G-Hilb}_{\mathbb{C}} \mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}/G$. Subsequently it has been considered in the case of finite small subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$ as well, see for example [Is02].

The relative tangent space of $\text{G-Hilb}_{\mathbb{C}} \mathbb{A}^n_{\mathbb{C}}$ over $\mathbb{A}^n_{\mathbb{C}}/G$ for points $\overline{I} \in E$ is given as

$$\operatorname{Hom}^G_{\overline{S}}(\overline{I}, \overline{S}/\overline{I})$$

Since a homomorphism of $\overline{S}$-modules $\overline{I} \to \overline{S}/\overline{I}$ is determined by the images of the generators of $\overline{I}$, one has an injective homomorphism of $\mathbb{C}$-vector spaces

$$\operatorname{Hom}^G_{\overline{S}}(\overline{I}, \overline{S}/\overline{I}) \to \operatorname{Hom}^G_{\mathbb{C}}(\overline{I}/\overline{\mathfrak{m}}\overline{I}, \overline{S}/\overline{I}) \tag{5}$$

In the 2-dimensional case, looking at the explicit structure of the fiber $\overline{S}$ of $\mathbb{A}^2_{\mathbb{C}} \to \mathbb{A}^2_{\mathbb{C}}/G$ over $\overline{O}$, one observes:

Observation 4.5. *The homomorphism* (5) *is an isomorphism for the cyclic groups* $G = \mu_r \subset \mathrm{SL}(2, \mathbb{C})$ *naturally operating on* $\mathbb{A}^2_{\mathbb{C}}$.

This relates the stratification discussed in [ItNm96], [ItNm99] to relative tangent spaces in the (A_n) cases. For the nonabelian finite subgroups of $\mathrm{SL}(2, \mathbb{C})$ some further considerations will be necessary – tangent spaces do not suffice to describe the spaces $\operatorname{Hom}^G_{\mathbb{C}}(\overline{I}/\overline{\mathfrak{m}}\overline{I}, \overline{S}/\overline{I})$. But (5) is surjective for finite abelian subgroups $G \subset \mathrm{GL}(2, \mathbb{C})$ and maybe in the abelian case under much more general circumstances.

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